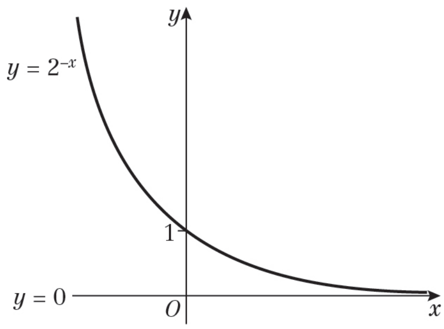
Chapter review 5

1 a $y = 2^{-x} = (2^{-1})^x = (\frac{1}{2})^x$



b $y = 5e^x - 1$

The graph is a translation by the vector $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ and a vertical stretch scale factor 5 of the graph $y = e^x$.

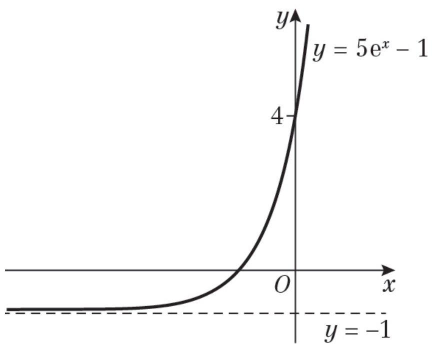
The graph crosses the y-axis when $x = 0$.

$y = 5 \times e^0 - 1$

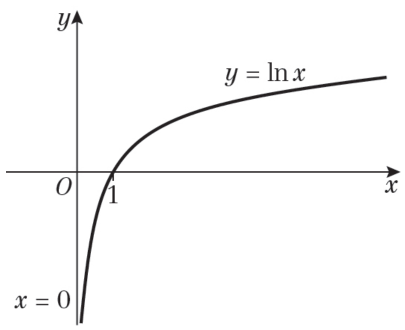
$y = 4$

The graph crosses the y-axis at (0, 4).

Asymptote is at $y = -1$.



c $y = \ln x$



2 a $\ln(p^2q) = \ln(p^2) + \ln(q)$
$= 2\ln(p) + \ln(q)$

b $\ln(pq) = 5$ and $\ln(p^2q) = 9$

$\ln(pq) = \ln(p) + \ln(q) = 5$ **(1)**

$\ln(p^2q) = 2\ln(p) + \ln(q) = 9$ **(2)**

Subtracting equation **(1)** from equation **(2)** gives:

$\ln(p) = 4$

Substituting into equation **(1)** gives:

$4 + \ln(q) = 5$

Therefore

$\ln(q) = 1$

3 a $y = e^{-x}$

$$\frac{dy}{dx} = -e^{-x}$$

b $y = e^{11x}$

$$\frac{dy}{dx} = 11e^{11x}$$

c $y = 6e^{5x}$

$$\frac{dy}{dx} = 5 \times 6e^{5x} = 30e^{5x}$$

4 a $\ln(2x-5) = 8 \quad$ (inverse of ln)

$2x - 5 = e^8 \quad (+5)$

$2x = e^8 + 5 \quad (\div 2)$

$$x = \frac{e^8 + 5}{2}$$

b $e^{4x} = 5 \quad$ (inverse of e)

$4x = \ln 5 \quad (\div 4)$

$$x = \frac{\ln 5}{4}$$

c $24 - e^{-2x} = 10 \quad (+e^{-2x})$

$24 = 10 + e^{-2x} \quad (-10)$

$14 = e^{-2x} \quad$ (inverse of e)

$\ln(14) = -2x \quad (\div -2)$

$-\frac{1}{2}\ln(14) = x$

$x = -\frac{1}{2}\ln(14)$

4 d $\ln(x)+\ln(x-3)=0$

$\ln(x(x-3))=0$

$x(x-3)=e^0$

$x(x-3)=1$

$x^2-3x-1=0$

$x=\frac{3\pm\sqrt{9+4}}{2}$

$=\frac{3\pm\sqrt{13}}{2}$

$=\frac{3+\sqrt{13}}{2}$

(x cannot be negative because of initial equation)

e $e^x+e^{-x}=2$

$e^x+\frac{1}{e^x}=2 \quad (\times\ e^x)$

$(e^x)^2+1=2e^x$

$(e^x)^2-2e^x+1=0$

$(e^x-1)^2=0$

$e^x=1$

$x=\ln 1=0$

f $\ln 2+\ln x=4$

$\ln 2x=4$

$2x=e^4$

$x=\frac{e^4}{2}$

5 $P=100+850e^{-\frac{t}{2}}$

a New price is when $t=0$

Substitute $t=0$ into $P=100+850e^{-\frac{t}{2}}$ to give:

$P=100+850e^{-\frac{0}{2}} \quad (e^0=1)$

$=100+850=950$

The new price is €950

5 b After 3 years $t=3$.

Substitute $t=3$ into $P=100+850e^{-\frac{t}{2}}$ to give:

$P=100+850e^{-\frac{3}{2}}=289.66$

Price after 3 years is €290 (to nearest €)

c It is worth less than €200 when $P<200$

Substitute $P=200$ into $P=100+850e^{-\frac{t}{2}}$ to give:

$200=100+850e^{-\frac{t}{2}}$

$100=850e^{-\frac{t}{2}}$

$\frac{100}{850}=e^{-\frac{t}{2}}$

$\ln\left(\frac{100}{850}\right)=-\frac{t}{2}$

$t=-2\ln\left(\frac{100}{850}\right)$

$t=4.28$

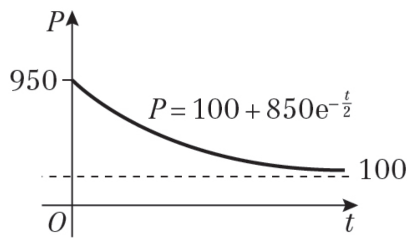
It is worth less than €200 after 4.28 years.

d As $t\to\infty$, $e^{-\frac{t}{2}}\to 0$

Hence, $P\to 100+850\times 0=100$

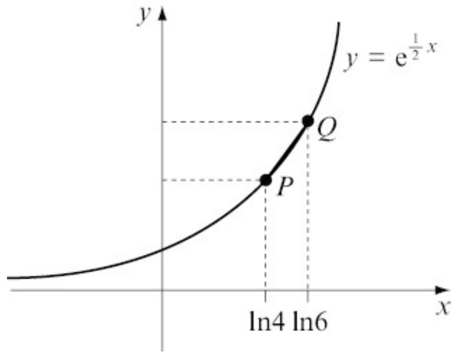
The computer will be worth €100 eventually.

e



f A good model. The computer will always be worth something.

6 a



Q has y-coordinate $e^{\frac{1}{2}\ln 16} = e^{\ln 16 \times \frac{1}{2}} = 16^{\frac{1}{2}} = 4$

P has y-coordinate $e^{\frac{1}{2}\ln 4} = e^{\ln 4 \times \frac{1}{2}} = 4^{\frac{1}{2}} = 2$

$$\text{Gradient of the line } PQ = \frac{\text{change in } y}{\text{change in } x} = \frac{4-2}{\ln 16 - \ln 4} = \frac{2}{\ln\left(\frac{16}{4}\right)} = \frac{2}{\ln 4}$$

Using $y = mx + c$, the equation of the line PQ is:

$$y = \frac{2}{\ln 4}x + c$$

$(\ln 4, 2)$ lies on the line so

$$y = \frac{2}{\ln 4}x + c$$

$$2 = 2 + c$$

$$c = 0$$

Equation of PQ is $y = \dfrac{2x}{\ln 4}$

b The line passes through the origin as $c = 0$.

c Length from $(\ln 4,\ 2)$ to $(\ln 16,\ 4)$ is

$$\sqrt{(\ln 16 - \ln 4)^2 + (4-2)^2}$$

$$= \sqrt{\left(\ln \tfrac{16}{4}\right)^2 + 2^2}$$

$$= \sqrt{(\ln 4)^2 + 4} = 2.43$$

7 a $T = 55e^{-\frac{t}{8}} + 20$

t is the time in minutes and time cannot be negative as you can't go back in time.

b The starting temperature of the cup of tea is when $t = 0$

$T = 55e^{-\frac{0}{8}} + 20 = 75°C$

c When $T = 50°C$

$$55e^{-\frac{t}{8}} + 20 = 50$$

$$55e^{-\frac{t}{8}} = 30$$

$$e^{-\frac{t}{8}} = \frac{30}{55}$$

$$\ln\left(e^{-\frac{t}{8}}\right) = \ln\left(\frac{30}{55}\right)$$

$$-\frac{t}{8} = \ln\left(\frac{30}{55}\right)$$

$$t = -8\ln\left(\frac{30}{55}\right)$$

$$= 4.849...$$

$$\approx 5 \text{ minutes}$$

d The exponential term will always be positive, so the overall temperature will be greater than 20°C.

8 a As $S = aV^b$

$\log S = \log (aV^b)$

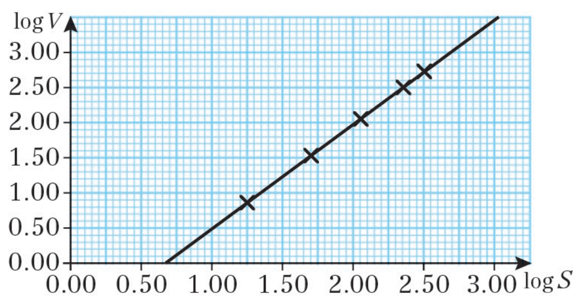
$\log S = \log a + \log (V^b)$

$\log S = \log a + b \log V$

b

$\log S$	1.26	1.70	2.05	2.35	2.50
$\log V$	0.86	1.53	2.05	2.49	2.72

c



8 d b is the gradient $= \dfrac{2.72-0.86}{2.5-1.26}$

$= \dfrac{1.86}{1.24} = 1.5$

Intercept $= \log a$
$\log a = -1.05$
$10^{-1.05} = a$
$a = 0.0891...$
$a \approx 0.09$

9 a The student goes wrong in line 2, where the subtraction should be a division (as in line 2 below).

b The full working should have looked like this:

$$\log_2 x - \tfrac{1}{2}\log_2(x+1) = 1$$
$$\log_2 x - \log_2\left((x+1)^{\frac{1}{2}}\right) = 1$$
$$\log_2 x - \log_2(\sqrt{x+1}) = 1$$
$$\log_2 \frac{x}{\sqrt{x+1}} = 1$$
$$\frac{x}{\sqrt{x+1}} = 2^1$$
$$x = 2\sqrt{x+1} \quad \text{(square)}$$
$$x^2 = 4x+4$$
$$x^2 - 4x - 4 = 0 \quad \text{(use quadratic formula)}$$
$$x = 2+2\sqrt{2}$$

($x \neq 2-2\sqrt{2}$ because log cannot take negative input values)

10 a The gradient is given by:

$$m = \frac{\log_{10} P_2 - \log_{10} P_1}{t_2 - t_1}$$
$$= \frac{2.2-2}{20-0}$$
$$= 0.01$$

The line crosses the vertical axis at (0, 2) therefore the equation of the line is:
$\log_{10} P = 0.01t + 2$

10 b $P = ab^t$

$$\log_{10}(P) = \log_{10}(ab^t)$$
$$= \log_{10}(a) + \log_{10}(b^t)$$
$$= \log_{10}(a) + t\log_{10}(b)$$

Comparing
$\log_{10}(P) = \log_{10}(a) + t\log_{10}(b)$
to
$\log_{10} P = 0.01t + 2$
gives:
$\log_{10}(a) = 2 \Rightarrow a = 100$

When $t = 0$, $P_0 = ab^0 = a$

Therefore, a represents the initial population of the colony.
When the population was first recorded, there were 100 ground-cuckoos.

c Comparing
$\log_{10}(P) = \log_{10}(a) + t\log_{10}(b)$
to
$\log_{10} P = 0.01t + 2$
gives:
$t\log_{10}(b) = 0.01t$

$$b = 10^{0.01}$$
$$= 1.023...$$
$$= 1.023 \text{ (3 d.p.)}$$

d Substituting $a = 100$ and $b = 10^{0.01}$ into $P = ab^t$ gives;

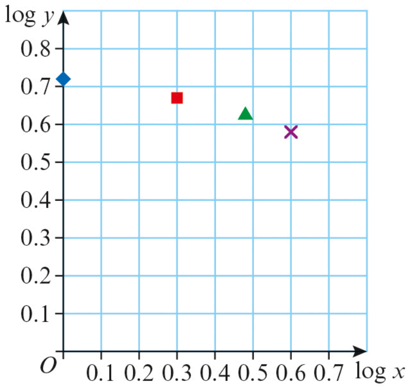
$P = 100 \times \left(10^{0.01}\right)^t = 100 \times 10^{0.01t}$

When $t = 30$:

$$P = 100 \times 10^{0.01(30)}$$
$$= 100 \times 10^{0.3}$$
$$= 199.526...$$
$$= 200 \text{ (3 s.f.)}$$

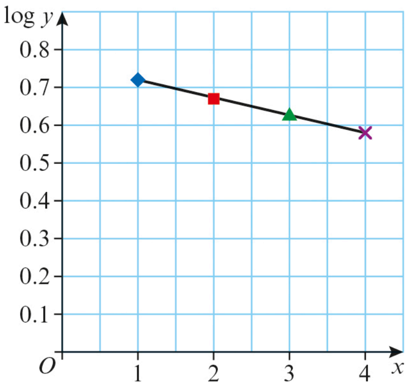
Challenge

$\log x$	0	0.30	0.48	0.60
$\log y$	0.72	0.67	0.63	0.58



The relationship between $\log x$ and $\log y$ is not linear so the relationship is perhaps $y = ax^n$

x	1	2	3	4
$\log y$	0.72	0.67	0.63	0.58



The second graph, $\log y$ against x, is a linear relationship so the relationship is of the form

$y = ab^x$

$\log y = \log (ab^x)$

$\log y = \log a + \log b^x$

$\log y = \log a + x \log b$

Intercept = 0.75

$\log a = 0.75$

$a = 10^{0.75} = 5.8$

$$\text{Gradient} = \frac{0.58 - 0.72}{4 - 1} = -\frac{0.14}{3} = -0.04666...$$

$\log b = -0.04666...$

$b = 10^{-0.04666...}$

$= 0.90$

So the formula is $y = 5.8 \times 0.9^x$